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Path Integral Approach to String Theory on AdS_3

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Abstract

Using the path integral approach, we discuss the correlation functions of the $SL(2, \mathbf{C})/SU(2)$ WZW model, which corresponds to the string theory on the Euclidean AdS_3 . We obtain the two- and three-point functions for generic primary fields in closed forms. By an appropriate change of the normalization of the primary fields, our results coincide with those by Teshner, which were obtained by using the bootstrap approach. The supergravity results are also obtained in the semi-classical limit.

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1 INTRODUCTION

The three-dimensional anti-de Sitter space (AdS_3) is a simple space-time with a constant negative curvature. From a group theoretical point of view, it is nothing but the $SL(2, \mathbf{R})$ group manifold. Because of this simplicity, it provides useful testing grounds for investigating strings in curved space-time and non-rational conformal field theories (see, for example, [1] and references therein). Furthermore, AdS_3 is closely related to various black hole geometries. This implies that the string theory on AdS_3 offers a key to the quantum theory of black holes.

Besides these features, AdS_3 gives the simplest example of the AdS/CFT correspondence [2]-[4]. This stimulated the recent studies of the string theories on AdS_3 and its Euclidean analog $SL(2, \mathbf{C})/SU(2) = H_3^+$. Since the S-dual configuration of the $D1/D5$ -brane system does not have the RR-field background, its near horizon geometry can be analyzed by the standard world-sheet technique, namely, the $SL(2, \mathbf{R})$ or H_3^+ WZW model [5]-[8]. In particular, the authors of [6, 7] showed that strings in the bulk of AdS_3 , or ‘short strings’, can be treated beyond the free field approach.

However, in spite of the importance of the $SL(2, \mathbf{R})$ and H_3^+ WZW models, it seems that these models are not yet completely understood. For a precise understanding, one needs to clarify the fundamental properties such as the true spectrum, modular invariance, fusion rules and unitarity. For recent progress and discussions on these issues, see [1],[9]-[15].

In this paper, we concentrate on the H_3^+ WZW model corresponding to the Euclidean AdS_3 . This model has been studied by several approaches, even before the AdS/CFT correspondence was proposed. The early one used path integral [16, 17], and the correlation functions were obtained for certain fields with non-negative half-integral $SL(2, \mathbf{C})$ spins [17]. The important fact is that the H_3^+ WZW model allows us the Lagrangian approach, which is impossible in the case of other WZW models. Later, for primary fields with generic spins, the correlation functions and fusion rules were discussed based on the symmetry and bootstrap [18, 19]. There are also arguments about the correlation functions using the free field realization of the \widehat{sl}_2 algebra [20, 10] and the supergravity approximation, e.g., [3, 4],[21]-[23]. Taking these into account, the H_3^+ WZW model seems to be more tractable than the $SL(2, \mathbf{R})$ WZW model. Furthermore, since the precise formulation of the AdS/CFT correspondence is given for the Euclidean AdS , H_3^+ seems to have a direct connection to this correspondence.

The aim of this paper is to discuss the correlation functions of the primary fields with generic spins using the path integral approach. In this approach, we can calculate the correlation functions directly by a somewhat familiar method. Moreover, this enables us to discuss beyond the ‘free field approximation’ [5, 10, 20]. In the following, we first argue that, by an appropriate definition of the correlation functions, their calculation is essentially reduced to that in [17]. We then discuss in detail the cases of two- and three-point functions and obtain them in closed forms. The supergravity calculation is recovered in the semi-classical limit. The results are also compared with those by Teschner [18, 19] and an exact agree-

ment is found after a change of normalizations. Thus, Teschner's approach and ours here are complementary to each other. However, because of the advantages mentioned above, further extensions of our approach may be possible. For example, we may be able to calculate the correlation functions of the energy-momentum tensor of the boundary CFT.

This paper is organized as follows. In section 2, we summarize the H_3^+ WZW model. In section 3, we review the discussion of [17] in some detail to make this paper self-contained. In section 4, the general formalism is given for the calculation of the correlation functions of generic primary fields. Section 5 is devoted to the calculation of the two- and three-point functions. In section 6, we compare our results with those from different approaches. We conclude with a brief discussion in section 7. Some useful integral formulas and properties of the Υ -function, which appear in section 6, are collected in appendix A and B, respectively.

2 H_3^+ WZW MODEL

We begin with a description of the conformal field theory whose target space is the Euclidean AdS_3 , namely, $SL(2, \mathbf{C})/SU(2) = H_3^+$ [5]-[7], [17]-[19]. After a brief summary, we introduce a spin 0 primary field, which appears in a later discussion.

2.1 Action and symmetry

An element g of H_3^+ is parametrized as

$$g = e^{\gamma\tau_+} e^{-\phi\tau_3} e^{\bar{\gamma}\tau_-} = \begin{pmatrix} e^{\phi}|\gamma|^2 + e^{-\phi} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix}, \quad (2.1)$$

where $\gamma^* = \bar{\gamma}$; and $\tau_{\pm} = (\tau_1 \pm i\tau_2)/2$ and τ_3 are Pauli matrices. The coset structure becomes manifest when g is written as

$$g = hh^{\dagger}, \quad h = \begin{pmatrix} e^{-\phi/2} & e^{\phi/2}\gamma \\ 0 & e^{\phi/2} \end{pmatrix} \in SL(2, \mathbf{C}), \quad (2.2)$$

because g is invariant under $h \rightarrow hu$ with $u \in SU(2)$. In this parametrization, the isometry of H_3^+ is

$$g \rightarrow g^A = AgA^{\dagger}, \quad A \in SL(2, \mathbf{C}). \quad (2.3)$$

The conformal field theory with the target space H_3^+ is described by the WZW action $S_{\text{WZW}}(g(z))$. Substituting (2.1) yields

$$S_{\text{WZW}} = \frac{k}{\pi} \int d^2z (\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma). \quad (2.4)$$

Here, k is the level of the WZW model and $d^2z = d\sigma_1 d\sigma_2$ with $z = \sigma_1 + i\sigma_2$. $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. The full theory is defined by this action and the invariant measure,

$$\mathcal{D}g = \mathcal{D}\phi \mathcal{D}(e^\phi \gamma) \mathcal{D}(e^\phi \bar{\gamma}). \quad (2.5)$$

The above action and measure have left and right affine symmetries $\widehat{SL}(2, \mathbf{C})_L \times \widehat{SL}(2, \mathbf{C})_R$, which act on $g(z)$ as $g(z) \rightarrow A(z)g(z)B^\dagger(z)$ with $A(z), B(z) \in SL(2, \mathbf{C})$. In order to keep g an element of H_3^+ , they need to be related to each other by $A(z) = B(z)$. Thus, the symmetry of the model is $\widehat{SL}(2, \mathbf{C}) \times \widehat{SL}(2, \mathbf{C})$. In particular, the global symmetry corresponds to a constant matrix A and given by (2.3). The currents of this global symmetry are represented by

$$J_0^- = \partial_\gamma, \quad J_0^3 = \gamma \partial_\gamma - \frac{1}{2} \partial_\phi, \quad J_0^+ = \gamma^2 \partial_\gamma - \gamma \partial_\phi - e^{-2\phi} \partial_{\bar{\gamma}}, \quad (2.6)$$

and similar expressions with bars.

The action (2.4) can be rewritten by introducing the auxiliary fields β and $\bar{\beta}$ [5]. The resultant action looks like an action for the free fields ϕ , β - γ and $\bar{\beta}$ - $\bar{\gamma}$, except for the term $\beta \bar{\beta} e^{-2\phi}$. This additional term drops out in the region $\phi \rightarrow \infty$, which corresponds to the boundary of H_3^+ . Thus, near to the boundary of H_3^+ the free field approach is applicable, but it is not completely clear to what extent one can use this approach in a generic region. Regarding this issue, see [6, 7, 10]. In our approach based on the full Lagrangian, we do not have such subtleties.

2.2 Primary fields

The primary fields of the model form the representations of the global $SL(2, \mathbf{C})$. These representations are well organized by introducing auxiliary coordinates (x, \bar{x}) [24]. They are interpreted as the coordinates of the boundary CFT in the AdS/CFT correspondence [6]. Using these, the spin j primary field is given by

$$\begin{aligned} \Phi_j(g(z), x) &= \left[(1, -x) g \begin{pmatrix} 1 \\ -\bar{x} \end{pmatrix} \right]^{2j} \\ &= \left(|\gamma(z) - x|^2 e^{\phi(z)} + e^{-\phi(z)} \right)^{2j}. \end{aligned} \quad (2.7)$$

Note that one cannot separate the left and right sectors in this expression. This is because the left and right symmetries are related to each other. By expanding Φ_j in terms of (x, \bar{x}) as

$$\Phi_j = \sum_{m, \bar{m}} x^{j-m} \bar{x}^{j-\bar{m}} \Phi_{m\bar{m}}^j, \quad (2.8)$$

one obtains primary fields with definite eigenvalues (m, \bar{m}) of (J_0^3, \bar{J}_0^3) . The range of m and \bar{m} depends on the value of j . For example, $\Phi_{1/2}$ is expanded as

$$\Phi_{1/2} = (|\gamma|^2 e^\phi + e^{-\phi}) - x(\bar{\gamma} e^\phi) - \bar{x}(\gamma e^\phi) + x\bar{x} e^\phi. \quad (2.9)$$

It is straightforward to check that the action of the $SL(2, \mathbf{C})$ currents on Φ_j gives

$$J_0^a \Phi_j(g, x) = -D^a \Phi(g, x), \quad (2.10)$$

where

$$D^- = \partial_x, \quad D^3 = x\partial_x - j, \quad D^+ = x^2\partial_x - 2jx. \quad (2.11)$$

In other words, Φ_j transforms under the global transformation (2.3) as

$$(R_A \Phi_j)(g, x) = \Phi_j(g^{A^{-1}}, x) = |cx + d|^{4j} \Phi_j(g, Ax) \quad (2.12)$$

with

$$Ax = \frac{ax + b}{cx + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C}). \quad (2.13)$$

In the discussions of the H_3^+ WZW model, various $SL(2, \mathbf{C})$ representations appear. An important class is called the principal continuous series. This class of representations is unitary, and has spin $j = -1/2 + i\rho$ ($\rho \in \mathbf{R}$).¹ The space of the square-integrable functions on H_3^+ is decomposed into these representations $\mathcal{H}_{-1/2+i\rho}$ [24]:

$$L^2(H_3^+, dg) \cong \int_{\rho>0}^{\otimes} d\rho \rho^2 \mathcal{H}_{-1/2+i\rho}. \quad (2.14)$$

A class of the representations with $j \leq -1/2$ appears in the discussion of the AdS/CFT correspondence [5]-[7]. This is an analog of the discrete series representations of $SL(2, \mathbf{R})$. When the spin is a non-negative half-integer, Φ_j is expanded into a finite sum of $\Phi_{m, \bar{m}}^j$ as in (2.9). This case appears in relation to the $SU(2)$ WZW model [17].

Since spin j is just the label of the second Casimir of the $SL(2, \mathbf{C})$, i.e., $-j(j+1)$, the representation with j and that with $-j-1$ are equivalent. This appears to be obvious for the principal continuous series because of the relation $-j-1 = j^*$. In general, the primary fields Φ_j and Φ_{-j-1} are classically related by

$$\Phi_j(g, x) = \frac{2j+1}{\pi} \int d^2y |x-y|^{4j} \Phi_{-j-1}(g, y). \quad (2.15)$$

However, this expression for a generic j may be modified at the values $j \in \mathbf{Z}/2$. This phenomena is called ‘resonance’ in [7].

¹ J_0^3 and \bar{J}_0^3 take $m = (ip + n)/2, \bar{m} = (ip - n)/2$ with $p \in \mathbf{R}, n \in \mathbf{Z}$. These are different from the corresponding representation of $SL(2, \mathbf{R})$ with the same j because m and \bar{m} are real in this case.

2.3 Spin 0 primary

It turns out that the spin 0 primary field appears in the definition of our correlation functions. An obvious ‘spin 0 primary field’ is just a constant Φ_0 . In addition to this, there exists a non-trivial spin 0 field. In fact, because of the formula

$$f(\gamma)\partial_\gamma^n\delta(\gamma-x)=\sum_{l=0}^n(-1)^l\binom{n}{l}\partial_x^lf(x)\partial_\gamma^{n-l}\delta(\gamma-x), \quad (2.16)$$

the operator

$$\widehat{\Phi}_0(g,x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}e^{-2(n+1)\phi}\partial_\gamma^n\partial_{\bar\gamma}^n\delta^2(\gamma-x) \quad (2.17)$$

satisfies (2.10) with $j=0$. Up to a coefficient and a constant term, this is also obtained by taking the limit $j\rightarrow 0$ in (2.15), namely,

$$\widehat{\Phi}_0\sim\int d^2x\,\Phi_{-1}(x). \quad (2.18)$$

In deriving this, we need the integral formula (A.2) in appendix A and

$$\lim_{\epsilon\rightarrow 0}\frac{|x|^{-2+2\epsilon}}{\Gamma(\epsilon)}=\pi\delta^2(x), \quad (2.19)$$

in [25]. This implies that the right-hand side of (2.15) with $j\rightarrow 0$ is different from Φ_0 in (2.7). This is an example of the ‘resonance’ mentioned above.

Since Φ_{-1} corresponds to the dimension $(1,1)$ operator of the boundary CFT, $\widehat{\Phi}_0$ is regarded as its vertex operator. Moreover, at least semi-classically Φ_{-1} satisfies [7]

$$\begin{aligned} \partial_{\bar z}\Phi_{-1} &= \partial_{\bar x}(\bar J\Phi_{-1}), \\ \bar J\Phi_{-1} &= \partial_{\bar z}\Lambda, \\ \Phi_{-1}\Phi_j &\sim (\text{regular terms in } z, \bar z), \end{aligned} \quad (2.20)$$

and similar equations with ∂_z, ∂_x and J . Here $\bar J(x,z)=2x\bar J^3(z)-\bar J^+(z)-x^2\bar J^-(z)$, $\bar J^a(z)$ are the \widehat{sl}_2 currents, and $\Lambda(g,x)$ is a certain function. J is defined similarly. The first two equations further imply that

$$\partial_{\bar z}\widehat{\Phi}_0=\partial_{\bar z}\int d^2x\partial_{\bar x}\Lambda\sim 0. \quad (2.21)$$

Thus, we see that $\widehat{\Phi}_0$ behaves as the identity (constant) operator on the world-sheet. Although this argument is based on the semi-classical analysis in [7], we will see that $\widehat{\Phi}_0$ actually behaves as the identity. This supports the argument in [7] conversely from the point of view of the full quantum theory. Note that the identity operator of the boundary theory is

$$I=\int d^2zJ\bar J\Phi_{-1}. \quad (2.22)$$

3 REVIEW OF PATH INTEGRAL APPROACH

The H_3^+ WZW model was discussed early in [16], and it was found that the functional integral of certain correlation functions can be performed. Later, such an argument was further developed by Gawędzki in relation to G/H coset models [17].

In the next section, we discuss the correlation functions of the primary fields Φ_j with generic j . We argue that the calculation of these correlation functions can be reduced to that of certain correlation functions with $j \in \mathbf{Z}_{\geq 0}/2$, which has been discussed by Gawędzki [17]. Thus, we first review his discussion. It is understood that all of the spins are non-negative half-integers in this section.

In the study of the H_3^+ WZW model, a difficulty often arises from the non-compactness of H_3^+ . In the path integral approach, this typically appears as the problem of zero-modes and requires a careful treatment of them in the definition of the correlation functions.

A naive definition of a correlation function may be

$$\langle \mathcal{O} \rangle \sim \int \mathcal{D}g e^{-S_{WZW}[g]} \mathcal{O}. \quad (3.1)$$

In a compact case such as $U(1)$, the zero-mode part in the functional integral $\int \mathcal{D}g$ picks up the invariant part of \mathcal{O} . However, in the non-compact case the zero-mode integral diverges generically. The prescription in [17] is: (i) choose \mathcal{O} which is already invariant under the global symmetry, and (ii) fix the zero-mode integral by inserting a delta function $\delta(g(z_0) - g_0)$ in the functional integral, where g_0 is a constant element of $SL(2, \mathbf{C})$. The delta function here maintains both the independence of z_0 and invariance under $\widehat{SL}(2, \mathbf{C})$: since \mathcal{O} is invariant under $SL(2, \mathbf{C})$, the insertion of $V^{-1} = V^{-1} \int dA \delta(g^A(z_0) - g_0)$ with $A \in SL(2, \mathbf{C})$ and V the volume of $SL(2, \mathbf{C})$ gives the delta function after the integration over A . Note that the integration over A splits into those over H_3^+ and $SU(2)$ since $g^A = (Ah)(Ah)^\dagger$. Hence, instead of (3.1), the correlation function is defined by

$$\langle \mathcal{O} \rangle = \frac{1}{Z_0} \int \mathcal{D}g \delta(g(z_0) - g_0) e^{-S_{WZW}[g]} \mathcal{O}, \quad (3.2)$$

with $Z_0 = \int \mathcal{D}g \delta(g(z_0) - g_0) e^{-S_{WZW}}$ and \mathcal{O} invariant under the global $SL(2, \mathbf{C})$. In the parametrization (2.1), the delta function takes the form

$$\begin{aligned} \delta(g(z_0) - g_0) &= \delta^2(e^{\phi(z_0)}(\gamma(z_0) - \gamma_0)) \delta(\phi(z_0) - \phi_0) \\ &= e^{-2\phi(z_0)} \delta^2(\gamma(z_0) - \gamma_0) \delta(\phi(z_0) - \phi_0). \end{aligned} \quad (3.3)$$

The simplest example of the correlation functions is the two-point function of the spin $j = 1/2$ field. In this case, the invariant operator is given by²

$$\begin{aligned} \mathcal{O}_2^G(j = 1/2) &= \text{Tr}(g(z_1)g^{-1}(z_2)) \\ &= e^{\phi(z_1) - \phi(z_2)} + e^{\phi(z_2) - \phi(z_1)} + e^{\phi(z_1) + \phi(z_2)} |\gamma(z_1) - \gamma(z_2)|^2. \end{aligned} \quad (3.4)$$

² From the geometrical point of view, this represents the distance in H_3^+ : $\frac{1}{2}\text{Tr}(g_1 g_2^{-1}) = u_{12} + 1 = \cosh \sigma_{12}$, where u_{12} and σ_{12} are the chordal and geodesic distances of H_3^+ , respectively.

Since the action is bi-linear in γ , the functional integral over γ is Gaussian and can be carried out. The propagator is then³

$$\langle \bar{\gamma}(z)\gamma(w) \rangle = \int \frac{d^2y}{\pi k} e^{-2\phi(y)} \left(\frac{1}{\bar{y}-\bar{z}} + \frac{1}{2}\bar{\partial}\sigma(y) \right) \left(\frac{1}{y-w} + \frac{1}{2}\partial\sigma(y) \right), \quad (3.5)$$

where σ is the conformal factor of the metric $g_{ab} = e^\sigma \delta_{ab}$ in the conformal gauge. This satisfies

$$-\frac{k}{\pi} \partial_{\bar{z}} e^{2\phi(z)} \partial_z \langle \bar{\gamma}(z)\gamma(w) \rangle = \delta^2(z-w) - \nu(z), \quad (3.6)$$

with

$$\nu(z) = \frac{1}{8\pi} \sqrt{g} R = -\frac{1}{2\pi} \partial \bar{\partial} \sigma, \quad (3.7)$$

and $\int \nu(z) = 1$. R is the world-sheet curvature. The σ -dependence in (3.5) disappears in the actual calculation, since it turns out that one needs only the σ -independent combination

$$\langle (\bar{\gamma}(z_1) - \bar{\gamma}(z_2)) (\gamma(z_3) - \gamma(z_4)) \rangle = \int \frac{d^2y}{\pi k} e^{-2\phi(y)} \frac{(\bar{z}_1 - \bar{z}_2)(z_3 - z_4)}{(\bar{y} - \bar{z}_1)(\bar{y} - \bar{z}_2)(y - z_3)(y - z_4)}. \quad (3.8)$$

When the world-sheet curvature is concentrated on $z = \hat{z}_0$, $\sigma(z)$ is given by $-4 \log |z - \hat{z}_0|$. In other words,

$$ds^2 = e^{\sigma(z)} |dz|^2 = \frac{|dz|^2}{|z - \hat{z}_0|^4} = \left| d \left(\frac{1}{z - \hat{z}_0} \right) \right|^2. \quad (3.9)$$

Then, the propagator further satisfies $\langle \bar{\gamma}(\hat{z}_0)\gamma(w) \rangle = 0$. This is consistent with the boundary condition following from the delta function (3.3) if $z_0 = \hat{z}_0$. Thus, this choice of the propagator implies that z_0 is the point of the support of the curvature. However, this fact is not important in the actual calculation, since the choice of \hat{z}_0 is arbitrary, and hence so is z_0 . In fact, the expressions of the correlators (without the σ -dependent part) turn out to be independent of z_0 and hence one need not necessarily take $z_0 \rightarrow \hat{z}_0$. This is understood as the remnant of the original $SL(2, \mathbf{C})$ invariance.

Taking the measure (2.5) into account, one finds that the γ -integration gives the Jacobian,

$$\text{Det}'^{-1} (e^{-\phi - \frac{1}{2}\sigma} \bar{\partial} e^{2\phi} \partial e^{-\phi - \frac{1}{2}\sigma}). \quad (3.10)$$

By the standard procedure, this Jacobian is found to be [26, 27]

$$\exp \left(\frac{1}{\pi} \int d^2z \, 2\partial\phi \bar{\partial}\phi + \frac{\phi}{4} \sqrt{g} R + \frac{1}{12} \partial\sigma \bar{\partial}\sigma \right), \quad (3.11)$$

³Here, we use γ for $\gamma - \gamma_0$ for simplicity. We use a similar kind of abuse of notation for ϕ in the following. Since we will calculate expectation values of quantities invariant under the global $SL(2, \mathbf{C})$, we do not have to be careful about such notations.

where we have dropped $\det'^{-1} \partial \bar{\partial}$, which is canceled by Z_0 .

Consequently, the resultant effective action for ϕ becomes

$$S_\phi = \frac{1}{\pi} \int d^2 z \left[(k-2) \partial \phi \bar{\partial} \phi - \frac{\phi}{4} \sqrt{g} R \right]. \quad (3.12)$$

Using this, the two-point function is written as

$$\begin{aligned} \langle \mathcal{O}_2^G(j=1/2) \rangle &\sim \int \mathcal{D}\phi \delta(\phi(z_0) - \phi_0) e^{-2\phi(z_0)} e^{-S_\phi} \\ &\times \left(e^{\phi(z_1) - \phi(z_2)} + e^{\phi(z_2) - \phi(z_1)} + e^{\phi(z_1) + \phi(z_2)} \langle |\gamma(z_1) - \gamma(z_2)|^2 \rangle \right). \end{aligned} \quad (3.13)$$

The ϕ -charge in this expression is neutral because the contribution from the anomaly term $\sqrt{g}R$ is canceled with that from $e^{-2\phi(z_0)}$.

The last ingredient to complete the calculation is the propagator of ϕ . The choice in [17] is

$$\begin{aligned} \langle \phi(z) \phi(w) \rangle &= -b^2 \log |z - w| + G_\sigma, \\ G_\sigma &= -\frac{1}{4} b^2 \left(\sigma(z) + \sigma(w) + \frac{1}{2\pi} \int d^2 z \sigma \partial \bar{\partial} \sigma \right), \end{aligned} \quad (3.14)$$

with

$$b^2 \equiv \frac{1}{k-2}. \quad (3.15)$$

This propagator satisfies

$$-\frac{2}{b^2 \pi} \partial_{\bar{z}} \partial_z \langle \phi(z) \phi(w) \rangle = \delta^2(z - w) - \nu(z), \quad (3.16)$$

and $\langle \phi(\hat{z}_0) \phi(w) \rangle = 0$. The latter is again consistent with the boundary condition if $z_0 = \hat{z}_0$. Note that, with this choice, the curvature term in (3.12) does not contribute to the following calculation, since $\int \langle \phi \phi \rangle \partial \bar{\partial} \sigma = 0$.

An important point here is that the ϕ -integration gives divergent factors through the self contraction of ϕ . We regularize this divergence by the point splitting method. Namely, we replace $\langle \phi(z) \phi(z) \rangle$ by

$$\langle \phi(z) \phi(z + \Delta z) \rangle = -b^2 \log \epsilon - \frac{b^2}{8\pi} \int \sigma \partial \bar{\partial} \sigma, \quad (3.17)$$

where ϵ is the infinitesimal UV cut-off and

$$\epsilon = \text{dist}(z, z + \Delta z) = e^{\frac{1}{2}\sigma(z)} |\Delta z|. \quad (3.18)$$

The strongest divergence comes from the term including $\langle |\gamma(z_1) - \gamma(z_2)|^2 \rangle$. In fact, it diverges as ϵ^{-5b^2} , since $e^{a\phi}$ in the correlator gives $\epsilon^{-a^2 b^2/2}$. This requires the multiplicative renormalization $Z_0 \rightarrow \epsilon^{-2b^2} Z_0$, which cancels the divergence from $e^{-2\phi(z_0)}$, and

$$\mathcal{O}_2^G(j=1/2) \rightarrow \epsilon^{-4\Delta_{1/2}} \mathcal{O}_2^G(j=1/2). \quad (3.19)$$

Here, Δ_j is the expected scaling dimension of the spin j field,

$$\Delta_j = -b^2 j(j+1). \quad (3.20)$$

Because of this renormalization, the first and second terms in (3.13) disappear. This is the simplest example of the general rule: for non-negative half-integral j , only the term with the highest power of γ survives the renormalization. This is confirmed by simple counting.

In all, omitting the factor including σ 's which have the support only at \hat{z}_0 , one arrives at

$$\begin{aligned} \langle \mathcal{O}_2^G(j=1/2) \rangle & \\ &= |(z_0 - z_1)(z_0 - z_2)|^{2b^2} |z_1 - z_2|^{2-b^2} \int d^2y |y - z_0|^{-4b^2} |(y - z_1)(y - z_2)|^{-2+2b^2}. \end{aligned} \quad (3.21)$$

By simple changes of variables, one confirms that this is independent of z_0 , as expected. The σ -dependence is found to be $\mathcal{A}_2(\sigma, z_1, z_2, \frac{1}{2}, \frac{1}{2})$, which is defined by [17]

$$\mathcal{A}_n(\sigma, z_a, j_a) = \exp \left(\frac{c}{24\pi} \int d^2z \partial \sigma \bar{\partial} \sigma - \sum_{a=1}^n \Delta_{j_a} \sigma(z_a) \right) \quad (3.22)$$

with

$$c = 2 + 1 + 6b^2 = \frac{3k}{k-2}. \quad (3.23)$$

This σ -dependence is canceled by the internal CFT and b - c ghosts when we consider the critical string theory.

An interesting consequence in this calculation is that the Coulomb-gas picture of the free field approach naturally appeared: the anomaly term $\sqrt{g}R$ corresponds to the charge at infinity (when $\hat{z}_0 \rightarrow \infty$) and the γ -propagator looks like the screening operator. Thus, the calculation here seems similar to that of the free field approach. However, the precise relationship does not seem to be completely clear.

The generalization of the above discussion to a generic $j \in \mathbf{Z}_{\geq 0}/2$ is straightforward. In such a case, the invariant combination of the two-point function is

$$\mathcal{O}_2^G(j) = P^{2j} \left[\text{Tr}(g(z_1)g^{-1}(z_2)) \right], \quad (3.24)$$

where $P^{2j}(x)$ is a polynomial of order $2j$ with coefficient 1 at x^{2j} . Repeating a similar procedure, one finds the renormalization

$$\mathcal{O}_2^G(j) \rightarrow \epsilon^{-4\Delta_j} \mathcal{O}_2^G(j), \quad (3.25)$$

and the term which survives the renormalization,

$$\tilde{\mathcal{O}}_2^G(j) = e^{2j(\phi(z_1)+\phi(z_2))} |\gamma(z_1) - \gamma(z_2)|^{4j}. \quad (3.26)$$

Finally, we consider the three-point function for spins j_1, j_2, j_3 with

$$j_1 + j_2 + j_3 \in \mathbf{Z}, \quad |j_1 - j_2| \leq j_3 \leq j_1 + j_2. \quad (3.27)$$

These conditions assure that j_{ab} defined by $j_{12} = j_1 + j_2 - j_3$ and similar expressions are also non-negative integers. In this case, the invariant combination is obtained by using an invariant tensor, the explicit form of which is found in [17]. Then, similarly to the above, one finds the renormalization factor $\epsilon^{-2(\Delta_{j_1} + \Delta_{j_2} + \Delta_{j_3})}$ and the relevant term after the renormalization,

$$\begin{aligned} \tilde{\mathcal{O}}_3^G(j_a) &= e^{2j_1\phi(z_1) + 2j_2\phi(z_2) + 2j_3\phi(z_3)} \\ &\times |\gamma(z_1) - \gamma(z_2)|^{2j_{12}} |\gamma(z_2) - \gamma(z_3)|^{2j_{23}} |\gamma(z_3) - \gamma(z_1)|^{2j_{31}}. \end{aligned} \quad (3.28)$$

The invariants (3.26) and (3.28) will appear in a later discussion.

4 CORRELATION FUNCTIONS OF PRIMARY FIELDS

In the previous section, we reviewed the calculation in [17] of certain correlation functions for non-negative half-integral spins. We now move on to the discussion of the correlation functions of the primary field $\Phi_j(g, x)$ with generic j .

4.1 Definition of correlation functions

In section 3, we saw that a careful treatment of zero-modes is necessary because of the non-compactness of H_3^+ . Taking this into account, we define the correlation function of Φ_j by

$$\left\langle \prod_{a=1}^n \Phi_{j_a}(g(z_a), x_a) \right\rangle = \frac{1}{Z} \int \mathcal{D}g e^{-S_{\text{WZW}}[g]} \hat{\Phi}_0 \mathcal{O}_n(g(z_a), j_a, x_a), \quad (4.1)$$

with $Z = \int \mathcal{D}g e^{-S_{\text{WZW}}[g]} \hat{\Phi}_0$. Here \mathcal{O}_n is the invariant part of $\prod_{a=1}^n \Phi_{j_a}(g(z_a), x_a)$, which will be determined in the next subsection. Since \mathcal{O}_n is invariant under the global symmetry and $\partial_\gamma = J_0^-$, we find that, in the correlator, $\hat{\Phi}_0$ is reduced to

$$\hat{\Phi}_0 \rightarrow e^{-2\phi(z_0)} \delta^2(\gamma(z_0) - x_0), \quad (4.2)$$

and hence $\mathcal{D}g \hat{\Phi}_0$ to

$$\mathcal{D}g \hat{\Phi}_0 \rightarrow \mathcal{D}(e^\phi \gamma) \mathcal{D}(e^\phi \bar{\gamma}) d\phi_0 \mathcal{D}'\phi \delta^2(\gamma(z_0) - x_0) e^{-2\phi(z_0)}. \quad (4.3)$$

Here, we have separated the measure of ϕ to its zero-mode part $d\phi_0$ and non-zero-mode part $\mathcal{D}'\phi$. Since the \mathcal{O}_n is invariant under the global $SL(2, \mathbf{C})$, the integrand in (4.1) does not depend on the zero-mode of ϕ . The divergent volume coming from the integration of ϕ_0 is

canceled by the same factor in Z . The $SL(2, \mathbf{C})$ invariance and the independence of z_0 and x_0 follow from the fact that $\hat{\Phi}_0$ behaves as the identity operator on the world-sheet.

Thus, the treatment of zero-modes here seems to be almost the same as that discussed in the previous section. In fact, we find that it is equivalent. In (3.3), an additional delta function $\delta(\phi(z_0) - \phi_0)$ was inserted instead of performing the zero-mode integral and dividing by its volume. This delta function imposed a boundary condition on $\phi(z)$. In turn, this boundary condition imposed the choice of the propagator (3.14) and the condition $z_0 \rightarrow \hat{z}_0$. However, such a difference does not matter. This is because (i) in the following calculation we use the same Green functions (3.5) and (3.14) (this is indeed a consistent choice), and (ii) though logically one should take $z_0 \rightarrow \hat{z}_0$ in the last step in the previous section, z_0 disappears in the actual calculation as discussed in the previous section.

Although our treatment of zero-modes and Gawędzki's are essentially equivalent, the use of $\hat{\Phi}_0$ has an advantage. Since $\hat{\Phi}_0$ transforms as a primary field under $SL(2, \mathbf{C})$, it is a conformal field. Thus, the conformal property of the correlation function is manifest. However $\delta(\phi(z_0) - \phi_0)$ does not transform properly under $SL(2, \mathbf{C})$, so its inclusion in the path integral makes the transformation property of the correlation functions apparently unclear.

The zero-mode integral is convergent for some correlation functions such as those of the primary fields in the principal continuous series. In such cases the prescription in eq.(4.1) coincides with the usual definition, because the zero-mode integral, which was fixed by the insertion of $\hat{\Phi}_0$, is essentially recovered by the projection defined in the next subsection.

4.2 $SL(2, \mathbf{C})$ projection

To proceed further, we need to determine the invariant part of $\prod_{a=1}^n \Phi_{j_a}(g(z_a), x_a)$. This is achieved by the following projection:⁴

$$\mathcal{O}_n(g(z_a), j_a, x_a) = \mathcal{N} \int_{SL(2, \mathbf{C})} dA \prod_{a=1}^n (R_A \Phi_{j_a})(g(z_a), x_a), \quad (4.4)$$

where \mathcal{N} is a normalization factor. Indeed, if the integral over $SL(2, \mathbf{C})$ is convergent, \mathcal{O}_n is invariant under $SL(2, \mathbf{C})$:

$$\mathcal{O}_n(g^A(z_a)) = \mathcal{O}_n(g(z_a)). \quad (4.5)$$

This projection can be performed explicitly. As the simplest example, let us first consider the $n = 2$ case. From (2.12), it follows that

$$\mathcal{O}_2(g(z_a)) = \mathcal{N} \int \frac{d^2 a d^2 c d^2 d}{|c|^2} |cx_1 + d|^{4j_1} |cx_2 + d|^{4j_2} \Phi_{j_1}(g(z_1), Ax_1) \Phi_{j_2}(g(z_2), Ax_2). \quad (4.6)$$

The change of variables $(a, c, d) \rightarrow (y, \lambda, w)$ with

$$y = Ax_1, \quad \lambda = cx_1 + d, \quad w = \frac{1}{\lambda c}, \quad (4.7)$$

⁴ Precisely, the integral is over $PSL(2, \mathbf{C})$ in the later calculations.

gives

$$\begin{aligned}
\mathcal{O}_2(g(z_a)) &= \mathcal{N} \int d^2 y d^2 \lambda d^2 c |\lambda|^{4j_1} |cx_{12} - \lambda|^{4j_2} \Phi_{j_1}(g(z_a), y) \Phi_{j_2}(g(z_2), y + \frac{x_{12}}{\lambda(cx_{12} - \lambda)}) \\
&\simeq \mathcal{N} |x_{12}|^{4j_2} \int d^2 y d^2 \lambda d^2 c |\lambda|^{4j_1} |c|^{4j_2} \Phi_{j_1}(g(z_1), y) \Phi_{j_2}(g(z_2), y + \frac{1}{\lambda c}) \\
&= \mathcal{N} |x_{12}|^{4j_2} \int d^2 y d^2 \lambda d^2 w |\lambda|^{-2+4(j_1-j_2)} |w|^{-4-4j_2} \Phi_{j_1}(g(z_1), y) \Phi_{j_2}(g(z_2), y + w),
\end{aligned} \tag{4.8}$$

were $x_{12} = x_1 - x_2$. We will use similar notations in the following. In going from the first line to the second line above, we have been sloppy about the treatment of the singular parameter region $x_{12} \rightarrow 0$. As we will show, a careful treatment of such a region gives an additional contribution to \mathcal{O}_2 . Let us define \mathcal{O}'_2 to be the quantity in the last line of eq.(4.8). By further renaming the variables, $\lambda = y_3$, $y = y_1$ and $y + w = y_2$, $\mathcal{O}'_2(g(z_a))$ becomes

$$\begin{aligned}
\mathcal{O}'_2(g(z_a)) &= \mathcal{N} |x_{12}|^{4j_2} \int \prod_{a=1}^3 d^2 y_a |y_3|^{-2+4(j_1-j_2)} |y_{12}|^{-4-4j_1} \Phi_{j_1}(g(z_1), y_1) \Phi_{j_2}(g(z_2), y_2) \\
&= \pi^2 \mathcal{N} i \delta(j_1 - j_2) |x_{12}|^{4j_2} \int d^2 y_1 d^2 y_2 |y_{12}|^{-4-4j_1} \Phi_{j_1}(g(z_1), y_1) \Phi_{j_1}(g(z_2), y_2).
\end{aligned} \tag{4.9}$$

To obtain the second line, the spins have to take the values of the principal continuous series $j_a = -1/2 + i\rho_a$ and $i\delta(j_1 - j_2)$ should be understood as $\delta(\rho_1 - \rho_2)$. For other values, the integral over y_3 is not well-defined. Thus, in such cases we ‘continue’ the expression of the second line to generic j . We will find that this prescription is consistent with the three-point function. In other words, we obtain the same result by (i) calculating the two-point functions for $j_a = -1/2 + i\rho_a$ from the three-point functions and (ii) continuing the final expression to generic spins. In any case, it is straightforward to check that (4.9) is invariant under the $SL(2, \mathbf{C})$ transformation. Note that the above integral is nothing but $\int d^2 x \Phi_{j_1}(g(z_1), x) \Phi_{-j_1-1}(g(z_2), x)$.

In (4.8), the change of variables becomes singular for $x_{12} \rightarrow 0$. In this case, another rescaling of the variables in (4.8) gives

$$\begin{aligned}
\mathcal{O}''_2 &= \mathcal{N} \int d^2 \lambda d^2 c |\lambda|^{4j_1} |cx_{12} - \lambda|^{4j_2} \int d^2 y \Phi_{j_1}(g(z_1), y) \Phi_{j_2}(g(z_2), y) \\
&= -\mathcal{N} \frac{\pi^4}{(2j_1 + 1)^2} i \delta(j_1 + j_2 + 1) \delta^2(x_{12}) \int d^2 y \Phi_{j_1}(g(z_1), y) \Phi_{j_2}(g(z_2), y),
\end{aligned} \tag{4.10}$$

where we have used (2.19). Such a contribution should be added to \mathcal{O}'_2 and we have $\mathcal{O}_2 = \mathcal{O}'_2 + \mathcal{O}''_2$.

Using the generators of $SL(2, \mathbf{C})$ in (2.11), one can show that the possible $SL(2, \mathbf{C})$ invariant combinations of x_a are only $i\delta(j_1 - j_2) |x_{12}|^{4j_1}$ and $i\delta(j_1 + j_2 + 1) \delta^2(x_{12})$ [18]. Thus we do not have any other invariants besides (4.9) and (4.10). The appearance of the contact

term (4.10) is one of the special features of the H_3^+ WZW model: it is possible since the left and right movers are combined from the beginning.

The projections in other cases are performed similarly to the $n = 2$ case. For generic spins, we then obtain

$$\begin{aligned}\mathcal{O}_3(g(z_a)) &= \frac{1}{2}\mathcal{N}\prod_{a<b}^3|x_{ab}|^{2j_{ab}}\int\prod_{a=1}^3d^2y_a\Phi_{j_a}(g(z_a),y_a)\prod_{a<b}^3|y_{ab}|^{-2-2j_{ab}}, \\ \mathcal{O}_4(g(z_a)) &= \frac{1}{2}\mathcal{N}\prod_{a<b}^4|x_{ab}|^{-\frac{2}{3}J+2(j_a+j_b)}\int_{y=x}\frac{d^2y_1d^2y_2d^2y_3}{|y_{12}|^2|y_{23}|^2|y_{31}|^2}\prod_{a=1}^4\Phi_{j_a}(y_a)\prod_{a<b}^4|y_{ab}|^{\frac{2}{3}J-2(j_a+j_b)}.\end{aligned}\quad (4.11)$$

Here, $J = \sum_{a=1}^4 j_a$; x and y are the cross-ratios $x = x_{13}x_{24}/x_{14}x_{23}$ and $y = y_{13}y_{24}/y_{14}y_{23}$ respectively; and

$$y_4 = \frac{y_1y_2(1-x) + y_3(y_1x - y_2)}{y_1 - y_2x + y_3(x-1)}.\quad (4.12)$$

The factor $1/2$ on the right hand side of (4.11) needs some explanation. For the change of the integration variables from (a, c, d) in (4.6) to y_a ($a = 1, 2, 3$), the Jacobian gives the factor $1/4$. However, because this change of variables is 2 to 1, we should multiply the integral by 2.

One can explicitly confirm that these are invariant under (2.12). For some special cases, possibly other invariants similar to \mathcal{O}_2' may appear. Note that \mathcal{O}_3 and \mathcal{O}_2 are related by

$$\lim_{j_3 \rightarrow -0} \mathcal{O}_3(g(z_a)) = \mathcal{O}_2(g(z_a)).\quad (4.13)$$

Here, j_3 should approach zero from the negative real axis so that the integral is convergent.

4.3 Continuation in j

Given the definition (4.1) and the invariants \mathcal{O}_n , we would like to calculate the correlation functions of generic primary fields Φ_j . In this paper, we obtain them from the correlation functions for $j \in \mathbf{Z}_{\geq 0}/2$ by continuing j to generic values. (We also use some consistency conditions to determine the two-point function.) The reason is two-fold. First, although Φ_j is expanded by polynomials in γ and $e^{\pm\phi}$ for $j \in \mathbf{Z}_{\geq 0}/2$, it becomes an infinite series for a generic j when expanded in γ and e^{ϕ} . In such a case, it is not clear if the classical expression (2.7) makes sense in the quantum theory. Second, it turns out that the explicit calculation is possible for $j \in \mathbf{Z}_{\geq 0}/2$, since it is reduced to that in section 3.

This prescription may be justified by defining Φ_j as

$$\begin{aligned}\Phi_j &= \frac{1}{\Gamma(-2j)}\int_0^\infty dt t^{-2j-1}\exp(-t\Phi_{\frac{1}{2}}) \\ &= \frac{1}{\Gamma(-2j)}\int_0^\infty dt t^{-2j-1}\sum_{n=0}^\infty \frac{(-1)^n}{n!}t^n\Phi_{\frac{n}{2}},\end{aligned}\quad (4.14)$$

for generic $j \notin \mathbf{Z}_{\geq 0}/2$. In this expression, we assume that all of the operators are regularized by the point splitting method. The renormalized operator is discussed later. From this definition, the invariant part of $\prod_{a=1}^n \Phi_{j_a}(g(z_a), x_a)$ is given by

$$\mathcal{O}_n(g(z_a), j_a, x_a) = \int_0^\infty \prod_{a=1}^n dt_a \frac{t^{-2j_a-1}}{\Gamma(-2j_a)} \sum_{m_a=0}^\infty \left(\prod_a \frac{(-1)^{m_a}}{m_a!} t_a^{m_a} \right) \mathcal{O}_n(g(z_a), \frac{m_a}{2}, x_a). \quad (4.15)$$

Furthermore, for an analytic function $f(x)$ its value at a generic point can be reconstructed from the data on the non-negative integers:

$$\begin{aligned} & \frac{1}{\Gamma(-x)} \int_0^\infty dt t^{-x-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^n f(n) \\ &= \frac{1}{\Gamma(-x)} \int_0^\infty dt t^{-x-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^n \int_0^\infty ds e^{-ns} \tilde{f}(s) \\ &= \int_0^\infty ds \tilde{f}(s) \frac{1}{\Gamma(-x)} \int_0^\infty dt t^{-x-1} e^{-te^{-s}} \\ &= \int_0^\infty ds e^{-xs} \tilde{f}(s) \\ &= f(x). \end{aligned} \quad (4.16)$$

Here, $\tilde{f}(s)$ is the inverse Laplace transform of $f(x)$. From (4.15) and (4.16), we find that $\langle \mathcal{O}_n(g(z_a), j_a, x_a) \rangle$ is obtained by the analytic continuation

$$\langle \mathcal{O}_n(g(z_a), j_a, x_a) \rangle = \left. \langle \mathcal{O}_n(g(z_a), \frac{m_a}{2}, x_a) \rangle \right|_{m_a=2j_a}. \quad (4.17)$$

This expression holds also when some of the spins j_a are non-negative half-integers. Thus the definition (4.14) indeed gives our prescription for the correlation functions. We will shortly see that (4.14) is also consistent with the renormalization.

If it is possible to calculate $\langle \mathcal{O}_n(g(z_a), m_a/2, x_a) \rangle$ for arbitrary $m_a \in \mathbf{Z}_{\geq 0}$, we could take (4.14) as the starting point of the discussion for generic j . However, we need to impose some conditions on m_a in the later calculation. Thus, our modest statement is that the continuation of the correlation functions, the definition of the primary fields Φ_j in (4.14) and the renormalization are all consistent.

In any case, what we need in the following is to calculate the correlation functions for the cases where the spins are non-negative half-integers and then continue the results to other cases. This is in the same spirit as that in [28] for the Liouville theory. However, we remark that in our case there exists a parameter region of j in which the correlation functions are actually calculable.

4.4 Renormalization

Once we focus on the case $j \in \mathbf{Z}_{\geq 0}/2$, the following calculation is carried out similarly to that in section 3. As noticed there, the renormalization picks up a term which has the

strongest divergence. Such a term is obtained by dropping $e^{-\phi}$ in Φ_j . For example, for \mathcal{O}'_2 in (4.9) the surviving term is

$$\begin{aligned}\tilde{\mathcal{O}}'_2(g(z_a)) &= \pi^2 \mathcal{N} |x_{12}|^{4j_2} i \delta(j_1 - j_2) e^{2j_1(\phi(z_1) + \phi(z_2))} \\ &\quad \times \int d^2 y_1 d^2 y_2 |y_{12}|^{-4-4j_1} |y_1 - \gamma_1|^{4j_1} |y_2 - \gamma_2|^{4j_1}.\end{aligned}\quad (4.18)$$

Here and in the following, it is understood that integrals such as the above are defined by the continuation from the parameter region in which they converge. By simple changes of integration variables, we further obtain

$$\begin{aligned}\tilde{\mathcal{O}}'_2(g(z_a)) &= \pi^2 \mathcal{N} a_2 i \delta(j_1 - j_2) \left(e^{\phi(z_1) + \phi(z_2)} |\gamma_{12}|^2 |x_{12}|^2 \right)^{2j_1} \\ &= \pi^2 \mathcal{N} a_2 i \delta(j_1 - j_2) |x_{12}|^{4j_1} \tilde{\mathcal{O}}_2^G(j_a),\end{aligned}\quad (4.19)$$

where

$$a_2 = \int d^2 y_1 d^2 y_2 |y_{12}|^{-4-4j_1} |y_1|^{4j_1} |y_2 - 1|^{4j_1}.\quad (4.20)$$

To compute a_2 , we should regularize the integral to define it:

$$\begin{aligned}a_2 &= \lim_{\epsilon \rightarrow 0} \int d^2 y_1 d^2 y_2 |y_{12}|^{-4-4j_1+\epsilon} |y_1|^{4j_1} |y_2 - 1|^{4j_1} \\ &= \lim_{\epsilon \rightarrow 0} \int d^2 y_1 |y_1 - 1|^{-4-4j_1+\epsilon} |y_1|^{4j_1} \int d^2 y_2 |y_2|^{-2+\epsilon} |y_2 - 1|^{4j_1} \\ &= -\frac{\pi^2}{(2j_1 + 1)^2}.\end{aligned}\quad (4.21)$$

Here, we have used the formula (A.2). The first and second integrals in the second line of (4.21) are zero and divergent, respectively, but the product gives a finite answer. Note that $\tilde{\mathcal{O}}'_2$ is the same up to a factor as the invariant combination $\tilde{\mathcal{O}}_2^G$ appeared in section 3. Thus the renormalization of \mathcal{O}'_2 is also given by (3.25). It is straightforward to check that the terms in (4.9) other than $\tilde{\mathcal{O}}'_2$ disappear after the renormalization.

If we simply apply the above argument to \mathcal{O}''_2 and drop the term $e^{-\phi}$ in Φ_j , we obtain

$$\begin{aligned}\tilde{\mathcal{O}}''_2 &= -\mathcal{N} \frac{\pi^4}{(2j_1 + 1)^2} i \delta(j_1 + j_2 + 1) \delta^2(x_{12}) e^{2j_1 \phi(z_1) + 2j_2 \phi(z_2)} \int d^2 y |y - \gamma_1|^{4j_1} |y - \gamma_2|^{-4-4j_1} \\ &= \mathcal{N} \frac{\pi^6}{(2j_1 + 1)^4} i \delta(j_1 + j_2 + 1) \delta^2(x_{12}) \delta^2(\gamma_{12}) e^{2j_1 \phi(z_1) + 2j_2 \phi(z_2)}.\end{aligned}\quad (4.22)$$

However, it is not clear if $\tilde{\mathcal{O}}''_2$ represents the correct contribution from \mathcal{O}''_2 after the renormalization: since \mathcal{O}''_2 always includes spins $j \notin \mathbf{Z}_{\geq 0}/2$, the above argument for $j \in \mathbf{Z}_{\geq 0}/2$ may not be valid. Here, we assume that \mathcal{O}''_2 is also renormalized by the multiplicative factor in (3.25). We do not use this expression in the actual calculation of $\langle \tilde{\mathcal{O}}''_2 \rangle$. Instead we determine it from consistency as in subsection 5.1.

For the three-point function with $j \in \mathbf{Z}_{\geq 0}/2$, the relevant term after the renormalization is obtained similarly to the case of \mathcal{O}'_2 :

$$\begin{aligned}
\tilde{\mathcal{O}}_3(g(z_a)) &= \frac{1}{2} \mathcal{N} \prod_{k < l}^3 |x_{kl}|^{2j_{kl}} \int \prod_{a=1}^3 d^2 y_a e^{2j_a \phi_a} |y_a - \gamma_a|^{4j_a} \prod_{a < b}^3 |y_a - y_b|^{-2-2j_{ab}} \\
&= \frac{1}{2} \mathcal{N} a_3 \prod_{a < b}^3 \left(e^{\phi_a + \phi_b} |\gamma_{ab}|^2 |x_{ab}|^2 \right)^{j_{ab}} \\
&= \frac{1}{2} \mathcal{N} a_3 \prod_{a < b}^3 |x_{ab}|^{2j_{ab}} \tilde{\mathcal{O}}_3^G.
\end{aligned} \tag{4.23}$$

Here, the coefficient a_3 is

$$\begin{aligned}
a_3 &= \int \prod_{a=1}^3 d^2 y_a |y_1|^{4j_1} |y_2 - 1|^{4j_2} \prod_{a < b}^3 |y_a - y_b|^{-2-2j_{ab}} \\
&= \pi^3 \Delta(-N-1) \prod_{a=1}^3 \frac{\Delta(2j_a - N)}{\Delta(-2j_a)},
\end{aligned} \tag{4.24}$$

with $N \equiv j_1 + j_2 + j_3$ and

$$\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{4.25}$$

We notice again that $\tilde{\mathcal{O}}_3$ is $\tilde{\mathcal{O}}_3^G$ up to a factor, and hence the renormalization is the same as that for $\tilde{\mathcal{O}}_3^G$.

These examples show that the divergence in the calculation is absorbed by the renormalization of the primary fields,

$$\Phi_{\frac{n}{2}}^{\text{ren}} = \epsilon^{-2\Delta_{n/2}} \Phi_{\frac{n}{2}}. \tag{4.26}$$

This expression makes sense even for a generic spin j through (4.14). To see this, we first rewrite the renormalization factor as

$$\epsilon^{-2\Delta_j} = \epsilon^{2b^2 j(j+1)} = \alpha^j \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\pi}} e^{-\lambda^2 + 2j\lambda\sqrt{\ln \alpha}}, \tag{4.27}$$

with $\alpha = \epsilon^{2b^2}$. Then, by rescaling the integration variable t , we find that

$$\epsilon^{-2\Delta_j} \Phi_j = \frac{1}{\Gamma(-2j)} \int_0^{\infty} dt t^{-2j-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n \epsilon^{-2\Delta_{n/2}} \Phi_{\frac{n}{2}}. \tag{4.28}$$

Thus, (4.26) indeed absorbs the divergence for a generic j . From now on, Φ_j should be understood to be the renormalized operator.

Such a prescription for renormalization of operators may be reliable for the cases in which the expression eq.(4.14) can be used for the calculation, for example, in calculating $\langle \mathcal{O}'_2 \rangle$. Here we assume the same prescription of renormalization for other cases. We will confirm its validity by checking the relations among various correlation functions and also by finding an agreement with the results obtained in other approaches.

5 TWO- AND THREE-POINT FUNCTIONS

We are now ready to go into the details of the calculation. In this section, we explicitly calculate the two- and three-point functions and obtain them in closed forms. According to the argument in the previous section, the spins are supposed to be non-negative half-integers in the calculation of $\langle \tilde{\mathcal{O}}'_2 \rangle$ and $\langle \tilde{\mathcal{O}}_3 \rangle$ until we arrive at the final expression. $\langle \mathcal{O}''_2 \rangle$ is determined by some consistency conditions. We then carry out the analytic continuation in j and obtain the results for generic spins.

5.1 Two-point functions

First, let us consider the two-point function

$$\langle \Phi_{j_1}(g(z_1), x_1) \Phi_{j_2}(g(z_2), x_2) \rangle = \langle \tilde{\mathcal{O}}'_2 \rangle + \langle \mathcal{O}''_2 \rangle, \quad (5.1)$$

with $\langle \tilde{\mathcal{O}}'_2 \rangle$ and $\langle \mathcal{O}''_2 \rangle$ given in (4.19) and (4.10), respectively. Following the procedure in section 3, $\langle \tilde{\mathcal{O}}_2^G \rangle$ in the first term is expressed by the integral [17]

$$\begin{aligned} \langle \tilde{\mathcal{O}}_2^G(j) \rangle &= \Gamma(2j+1) |(z_0 - z_1)(z_0 - z_2)|^{4jb^2} |z_{12}|^{4j-4j^2b^2} \\ &\times \int \prod_{a=1}^{2j} \left(\frac{d^2 y_a}{\pi k} |y_a - z_0|^{-4b^2} |(y_a - z_1)(y_a - z_2)|^{-2+4jb^2} \right) \prod_{a < b}^{2j} |y_a - y_b|^{-4b^2}. \end{aligned} \quad (5.2)$$

By making simple changes of variables⁵ and using the Dotsenko-Fateev formula (A.1) [33], we obtain

$$\langle \tilde{\mathcal{O}}_2^G(j) \rangle = C_2(j) |z_{12}|^{4j(j+1)b^2}, \quad (5.3)$$

with

$$\begin{aligned} C_2(j) &= \Gamma(2j+1) \int \prod_{a=1}^{2j} \left(\frac{d^2 y_a}{\pi k} |y_a(y_a - 1)|^{-2+4jb^2} \right) \prod_{a < b}^{2j} |y_a - y_b|^{-4b^2} \\ &= -kb^4(2j+1)^2 \left(\frac{\Delta(b^2)}{k} \right)^{2j+1} \Delta(-(2j+1)b^2). \end{aligned} \quad (5.4)$$

$C_2(j)$ satisfies $C_2(0) = 1$ and $\langle \tilde{\mathcal{O}}_2^G(j) \rangle$ is independent of z_0 as it should be. This has poles at $2j+1 = n(k-2)$ with positive integers n . In particular, the first one from $n = 1$ corresponds to the convergence condition discussed in [17], which was associated with the fusion rule $j \leq \tilde{k}/2 \equiv (k-4)/2$ of the $SU(2)$ WZW model. Combining $C_2(j)$ with a_2 yields

$$B(j) \equiv \pi^2 \mathcal{N} a_2 C_2(j) = \mathcal{N} \pi^4 k b^4 \left(\frac{\Delta(b^2)}{k} \right)^{2j+1} \Delta(-(2j+1)b^2). \quad (5.5)$$

⁵ One may take $z_0 = \infty$ here but this is not necessary.

Next, we turn to the contribution from $\langle \mathcal{O}_2'' \rangle$. Since this term always has $j \notin \mathbf{Z}_{\geq 0}/2$ and may contain distributions such as $\delta^2(\gamma_{12})$ in (4.22), we do not know how to calculate its expectation value in our formalism. However, it is determined by the consistency as follows.

First, let us recall that Φ_j and Φ_{-j-1} are classically related by (2.15). The structure of the integral transformation is almost completely fixed by the $SL(2, \mathbf{C})$ symmetry. Here, we allow that the coefficient in front of the integral changes in the quantum theory as

$$\Phi_j(g, x) = R(j) \int d^2 y |x - y|^{4j} \Phi_{-j-1}(g, y). \quad (5.6)$$

$R(j)$ is the reflection coefficient. Repeating the above transformation twice gives

$$R(j)R(-j-1) = -\frac{(2j+1)^2}{\pi^2}. \quad (5.7)$$

Next, by introducing $A(j)$ to denote the coefficient in $\langle \mathcal{O}_2'' \rangle$, we rewrite (5.1) as

$$\begin{aligned} & \langle \Phi_{j_1}(g(z_1), x_1) \Phi_{j_2}(g(z_2), x_2) \rangle \\ &= |z_{12}|^{4b^2 j_1(j_1+1)} \left[A(j_1) i \delta(j_1 + j_2 + 1) \delta^2(x_{12}) + B(j_1) i \delta(j_1 - j_2) |x_{12}|^{4j_1} \right]. \end{aligned} \quad (5.8)$$

Substituting (5.6) into Φ_{j_1} , we obtain

$$\begin{aligned} A(j) &= -\frac{\pi^2}{(2j+1)^2} R(j) B(-j-1), \\ B(j) &= R(j) A(-j-1). \end{aligned} \quad (5.9)$$

Further substitution of (5.6) into Φ_{j_2} gives

$$\begin{aligned} A(j) &= A(-j-1), \\ B(j) &= -\frac{\pi^2}{(2j+1)^2} R^2(j) B(-j-1). \end{aligned} \quad (5.10)$$

Together with the result of $B(j)$, these determine $A(j)$ and $R(j)$:

$$\begin{aligned} A(j) &= -\mathcal{N} \frac{\pi^5 k b^2}{(2j+1)^2}, \\ R(j) &= -\frac{(2j+1)^2 b^2}{\pi} \left(\frac{\Delta(b^2)}{k} \right)^{2j+1} \Delta(-(2j+1)b^2). \end{aligned} \quad (5.11)$$

Since the consistency conditions (5.9) and (5.10) are invariant under $(A(j), B(j), R(j)) \rightarrow (-A(j), B(j), -R(j))$, there is an ambiguity in the sign of $A(j)$ and $R(j)$ for a given $B(j)$. This sign is fixed by demanding that $R(j)$ is reduced to its classical value $(2j+1)/\pi$ in the limit $k \rightarrow \infty$. This completes the calculation of the two-point function.

5.2 Three-point functions

Let us move on to the discussion of the three-point function. In the previous section, we argued that for generic j the three-point function is given by

$$\begin{aligned} & \langle \Phi_{j_1}(g(z_1), x_1) \Phi_{j_2}(g(z_2), x_2) \Phi_{j_3}(g(z_3), x_3) \rangle \\ &= \langle \tilde{\mathcal{O}}_3(g(z_a), x_a) \rangle \equiv \frac{1}{2} \mathcal{N} a_3 \prod_{a < b}^3 |x_{ab}|^{2j_{ab}} G_3(j_a, z_a), \end{aligned} \quad (5.12)$$

with $G_3 \equiv \langle \tilde{\mathcal{O}}_3^G \rangle$. Following the procedure in section 3, for the spins satisfying (3.27), G_3 is given by the integral

$$\begin{aligned} G_3 &= \prod_{a < b}^3 |z_a - z_b|^{-4j_a j_b b^2 + 2j_{ab}} \prod_{a=1}^3 |z_0 - z_a|^{4j_a b^2} \int \prod_{a=1}^N \frac{d^2 y_a}{\pi k} |y_a - z_0|^{-4b^2} \\ &\times \prod_{b=1}^3 |y_a - z_b|^{4j_a b^2} \prod_{a < b}^N |y_a - y_b|^{-4b^2} \sum_{\sigma \in S_N} \prod_{a \leq 2j_1} [(y_a - z_1)(\bar{y}_{\sigma(a)} - \bar{z}_1)]^{-1} \\ &\times \prod_{a \leq j_{12} \text{ or } a > 2j_1} [(y_a - z_2)(\bar{y}_{\sigma(a)} - \bar{z}_2)]^{-1} \prod_{a > j_{12}} [(y_a - z_3)(\bar{y}_{\sigma(a)} - \bar{z}_3)]^{-1}, \end{aligned} \quad (5.13)$$

where S_N stands for the permutations of $N = j_1 + j_2 + j_3$ elements. Note that $2j_2 \geq j_{12}$. By some changes of variables, this is brought into the form

$$G_3 = \prod_{a < b}^3 |z_{ab}|^{2\Delta_{ab}} C(j_1, j_2, j_3, \xi), \quad (5.14)$$

where ξ is the cross-ratio

$$\xi = \frac{z_{01} z_{23}}{z_{03} z_{21}}, \quad (5.15)$$

and Δ_{ab} are given by

$$\Delta_{12} = \Delta_{j_3} - \Delta_{j_1} - \Delta_{j_2} = b^2 [j_{12}(N+1) - 2j_1 j_2], \quad (5.16)$$

and similar expressions. The coefficient $C(j_1, j_2, j_3, \xi)$ is roughly speaking a kind of a four-point function,

$$\int e^{-S_{WZW}} \Phi_{j_1}(0) \Phi_{j_2}(1) \Phi_{j_3}(\infty) \hat{\Phi}_0(\xi), \quad (5.17)$$

and we can obtain

$$\begin{aligned} & C(j_1, j_2, j_3, \xi) \\ &= |\xi|^{4b^2 j_1} |1 - \xi|^{4b^2 j_2} \int \prod_{a=1}^N \frac{d^2 y_a}{\pi k} |y_a|^{4b^2 j_1} |y_a - 1|^{4b^2 j_2} |y_a - \xi|^{-4b^2} \prod_{a < b} |y_a - y_b|^{-4b^2} \\ &\times \sum_{\sigma \in S_N} \prod_{a \leq 2j_1} \frac{1}{y_a \bar{y}_{\sigma(a)}} \prod_{a \leq j_{12}} \frac{1}{(y_a - 1)(\bar{y}_{\sigma(a)} - 1)} \prod_{a > 2j_1} \frac{1}{(y_a - \xi)(\bar{y}_{\sigma(a)} - \bar{\xi})}. \end{aligned} \quad (5.18)$$

Since j_a ($a = 1, 2, 3$) were on an equal footing originally, different changes of variables in G_3 give expressions in which j_a are permuted:

$$C(j_1, j_2, j_3, \xi) = C(j_2, j_1, j_3, 1 - \xi) = C(j_3, j_2, j_1, \frac{1}{\xi}). \quad (5.19)$$

Since Φ_j are conformal fields, their correlation function factorizes into the holomorphic and anti-holomorphic parts. Therefore, if $C(j_a, \xi)$ has no singularity in ξ , it is an entire function on \mathbf{CP}^1 , a constant. The possible singularities of $C(j_a, \xi)$ are located at the insertion points of other operators, i.e., $\xi = 0, 1, \infty$. From the relation (5.19), the existence of the limit

$$\lim_{\xi \rightarrow 0} C(j_1, j_2, j_3, \xi) \equiv C_3(j_1, j_2, j_3), \quad (5.20)$$

ensures that $C(j_a, \xi)$ is independent of ξ , as it should be. We show this by an explicit calculation.

For this purpose, we make the rescalings of variables

$$y_a = \xi w_a, \quad (a = 1, \dots, 2j_1). \quad (5.21)$$

We can then take the limit $\xi \rightarrow 0$ and find that many terms drop out. Consequently, we obtain

$$\begin{aligned} C_3(j_a) &= \Gamma(2j_1 + 1)\Gamma(j_{23} + 1) \int \prod_{a \leq 2j_1} \frac{d^2 w_a}{\pi k} |w_a|^{4b^2 j_1 - 2} |w_a - 1|^{-4b^2} \prod_{a < b \leq 2j_1} |w_a - w_b|^{-4b^2} \\ &\quad \times \int \prod_{a > 2j_1} \frac{d^2 y_a}{\pi k} |y_a|^{-4(j_1+1)b^2} |y_a - 1|^{4j_2 b^2 - 2} \prod_{2j_1 < a < b} |y_a - y_b|^{-4b^2}. \end{aligned} \quad (5.22)$$

Here, we have used the following equations:

$$\begin{aligned} &\lim_{\xi \rightarrow 0} \prod_{a=1}^N d^2 y_a \sum_{\sigma \in S_N} \prod_{a \leq 2j_1} \frac{1}{y_a \bar{y}_{\sigma(a)}} \prod_{a \leq j_{12}} \frac{1}{(y_a - 1)(\bar{y}_{\sigma(a)} - 1)} \prod_{a > 2j_1} \frac{1}{(y_a - 1)(\bar{y}_{\sigma(a)} - 1)} \\ &= \prod_{a \leq 2j_1} d^2 w_a \prod_{a > 2j_1} d^2 y_a \sum_{\sigma \in S_{2j_1}} \sum_{\tau \in S_{j_{23}}} \prod_{a \leq 2j_1} \frac{1}{w_a \bar{w}_{\sigma(a)}} \prod_{a > 2j_1} \frac{1}{(y_a - 1)(\bar{y}_{\tau(a)} - 1)} \\ &= \Gamma(2j_1 + 1)\Gamma(j_{23} + 1) \prod_{a \leq 2j_1} d^2 w_a |w_a|^{-2} \prod_{a > 2j_1} d^2 y_a |y_a - 1|^{-2}. \end{aligned} \quad (5.23)$$

From (5.22), we find that $C_3(j_a)$ is factorized into a product of two Dotsenko-Fateev integrals in (A.1):

$$\begin{aligned} C_3(j_a) &= \frac{1}{(\pi k)^N} \Gamma(2j_1 + 1)\Gamma(j_{23} + 1) \\ &\quad \times J_{2j_1}(2b^2 j_1 - 1, -2b^2, b^2) J_{j_{23}}(-2b^2 j_1 - 2b^2, 2b^2 j_2 - 1, b^2). \end{aligned} \quad (5.24)$$

The first integral is evaluated to be

$$J_{2j_1}(2b^2 j_1 - 1, -2b^2, b^2) = \frac{(\pi \Delta(b^2))^{2j_1+1}}{\pi \Gamma(2j_1 + 1) \Delta((2j_1 + 1)b^2)}. \quad (5.25)$$

The second integral takes the form $J_{j_{23}} \sim \prod_{n=1}^{j_{23}} [\Delta(f_1(n))\Delta(f_2(n))\dots]$ with certain functions $f_a(n)$. Although, as discussed in section 4, we would like to analytically continue the final expression in term of j_a , it appears difficult to do so in this form. However, this is achieved with the help of the Υ -function introduced in [34] (see also [35]). We have collected the definition and basic properties of $\Upsilon(x)$ in appendix B. To use $\Upsilon(x)$, we first rewrite $J_{j_{23}}$ using $I_m(\alpha_a)$ defined in (B.4):

$$J_{j_{23}}(-2b^2j_1 - 2b^2, 2b^2j_2 - 1, b^2) = I_{j_{23}}(j_1b + b, -j_2b + \frac{1}{2b}, -j_3b + \frac{1}{2b}). \quad (5.26)$$

By further making use of the relation between $\Upsilon(x)$ and I_m , (B.6), we arrive at

$$C_3(j_a) = kb \left(\frac{1}{k} b^{-2b^2} \Delta(b^2) \right)^{N+1} \frac{\Upsilon'(0)}{\Upsilon((N+2)b)} \prod_{a=1}^3 \frac{\Upsilon((2j_a+1)b)}{\Upsilon((N-2j_a+1)b)}, \quad (5.27)$$

where $\Upsilon'(x) = d\Upsilon/dx$. This is symmetric with respect to j_a , though the expressions in the intermediate stage were not. Since $\Upsilon(x)$ is analytic in x , we can continue the above expression to that for arbitrary j_a . Finally, putting everything together, we obtain the expression of the three-point function,

$$\left\langle \prod_{a=1}^3 \Phi_{j_a}(g(z_a), x_a) \right\rangle = D(j_a) \prod_{a < b} |x_{ab}|^{2j_{ab}} |z_{ab}|^{2\Delta_{ab}}, \quad (5.28)$$

with

$$\begin{aligned} D(j_a) &\equiv \frac{1}{2} \mathcal{N}_{a3} C_3(j_a) \\ &= \frac{1}{2} \mathcal{N} \pi^3 k b^4 \left(\frac{1}{k} b^{-2b^2} \Delta(b^2) \right)^{N+1} \frac{\Upsilon'(0)}{\Upsilon(-(N+1)b)} \prod_{a=1}^3 \frac{\Upsilon(-2j_a b)}{\Upsilon((2j_a - N)b)}. \end{aligned} \quad (5.29)$$

The structure of the poles of the three-point function is important to consider the fusion rules [18, 19]. It is read off from the zeros of $\Upsilon(x)$ given in (B.3). For generic j_a , $D(j_a)$ has poles at

$$j_{ab}, N+1 = m + nb^{-2}, -(m+1) - (n+1)b^{-2}, \quad m, n \in \mathbf{Z}_{\geq 0}. \quad (5.30)$$

However, for example, for $j_a \in \mathbf{Z}_{\geq 0}/2$, many poles are canceled with the zeros from the numerator. This is also confirmed by noting that in this case $\Upsilon(x)$ is reduced to the Dotsenko-Fateev integral given by a product of $\Delta(x)$. Incidentally, the pole at $N = k - 3 = \tilde{k} + 1$ corresponds to the convergence condition discussed in [17], which was associated to the three-point fusion rule of the $SU(2)$ WZW model, $j_1 + j_2 + j_3 \leq \tilde{k}$.

Finally, let us check the consistency of our calculation. As the simplest check, we see that $C_3(j, j, 0) = C_2(j)$. Furthermore, for $j_{1,2} = -1/2 + i\rho_{1,2}$ and $j_3 \rightarrow -0$, the three-point function is reduced exactly to the two-point function:

$$\begin{aligned} &\lim_{j_3 \rightarrow -0} D(j_a) \prod_{a < b} |x_{ab}|^{2j_{ab}} |z_{ab}|^{2\Delta_{ab}} \\ &= |z_{12}|^{-4\Delta_{j_1}} \left[A(j_1) i \delta(j_1 + j_2 + 1) \delta^2(x_{12}) + B(j_1) i \delta(j_1 - j_2) |x_{12}|^{4j_1} \right]. \end{aligned} \quad (5.31)$$

To derive this relation, we need to take the limit carefully so that we do not miss the distributions [19]. Here, the factor a_3 , which comes from the $SL(2, \mathbf{C})$ projection, gives the delta functions of j 's.

6 COMPARISON WITH OTHER APPROACHES

In the previous section, we obtained the closed forms of the two- and three-point functions (5.8) and (5.28), which are valid for generic spins. Now let us compare these with the results obtained by other approaches.

6.1 Supergravity approximation

In the discussion of the AdS/CFT correspondence, the various correlation functions have been calculated in the supergravity approximation, e.g., [3, 4],[21]-[23]. The supergravity approximation for the correlation functions of Φ_j is nothing but the zero-mode approximation from the point of view of the H_3^+ WZW model [18].

Note that Φ_j coincides with the boundary-to-bulk propagator on AdS_3 , i.e., Φ_j satisfies

$$\begin{aligned} (\Delta - m^2)\Phi_j(g, x) &= 0, \\ \lim_{\phi \rightarrow \infty} \Phi_j(g, x) &= -\frac{\pi}{2j+1} e^{-2(j+1)\phi} \delta^2(\gamma - x), \end{aligned} \quad (6.1)$$

where $\Delta = -4\eta_{ab}J_0^a J_0^b/k$ is the Laplacian on AdS_3 and the mass m and spin j are related by

$$j = -\frac{1}{2} - \frac{1}{2}\sqrt{1 + km^2}. \quad (6.2)$$

Since the two-point function in the supergravity calculation is not the same object as ours, we will not make a direct comparison. In the case of the three-point function, the supergravity result [18, 22] is obtained by following the so-called GKPW-prescription [3, 4]:

$$\int e^{2\phi} d\phi d^2\gamma \prod_{a=1}^3 \Phi_{j_a}(g, x_a) = D(j_a)^{SG} \prod_{a < b} |x_{ab}|^{2j_{ab}}, \quad (6.3)$$

with

$$D(j_a)^{SG} = \frac{\pi}{2} \Gamma(-N-1) \prod_{a=1}^3 \frac{\Gamma(2j_a - N)}{\Gamma(-2j_a)}. \quad (6.4)$$

In our world-sheet calculation, the supergravity limit corresponds to $\alpha' \rightarrow 0$ or $k \rightarrow \infty$. Since $\Upsilon(x)$ appears singular in this limit, i.e., $b \rightarrow 0$ (see (B.1)), it is useful to go back to the expression using J_m . We then find that in this limit our calculation is reduced to the supergravity approximation as expected:

$$\lim_{k \rightarrow \infty} D(j_a) = \mathcal{N} \pi^2 D(j_a)^{SG}. \quad (6.5)$$

From (6.4), we see that the basic structure of $D(j_a)^{SG}$, such as the location of the poles, is encoded in the coefficient a_3 in (4.24). This is because the zero-mode integral in (6.3) is essentially the same as the integral in the $SL(2, \mathbf{C})$ projection in (4.4). However, the precise connection is not yet clear. The coefficient in (6.5) may indicate a difference of these two integrals.

6.2 Bootstrap approach

Next, we turn to the other approach to the quantum theory. In [18, 19], the correlation functions for generic spins are discussed based on the symmetry and bootstrap. In particular, the three-point function is obtained as the solution to the functional relation derived from the crossing symmetry.

Since the normalizations in [18] and [19] are different, we first consider the result in [19]. The three-point function in [19] corresponding to our $D(j_a)$ is⁶

$$D(j_a)^T = -\frac{1}{2\pi^3 b} \left(\frac{\pi b^{2b^2-2}}{\Delta(b^2)} \right)^{N+2} \frac{\Upsilon'(0)}{\Upsilon(-(N+1)b)} \prod_{a=1}^3 \frac{\Upsilon(-(2j_a+1)b)}{\Upsilon((2j_a-N)b)}. \quad (6.6)$$

The two-point function is obtained by taking the limit $j_3 \rightarrow -0$ with $j_a = -1/2 + i\rho_a$ ($a = 1, 2$) as in (5.31):

$$\lim_{j_3 \rightarrow -0} D(j_a)^T \prod_{a < b}^3 |x_{ab}|^{2j_{ab}} = i\delta(j_1 - j_2) B(j_1)^T |x_{12}|^{4j_1} + i\delta(j_1 + j_2 + 1) \delta^2(x_{12}), \quad (6.7)$$

where

$$B(j)^T = \frac{1}{\pi b^2} \left(\frac{\pi}{b^2 \Delta(b^2)} \right)^{2j+1} \Delta(-(2j+1)b^2)^{-1}. \quad (6.8)$$

Once the above expression is obtained, one can continue it to generic j . In this expression, the quantity corresponding to our $A(j)$ is $A(j)^T = 1$, and the reflection coefficient in [19] is given by $R(j)^T = B(j)^T$. We can check that these $A(j)^T$, $B(j)^T$ and $R(j)^T$ satisfy the consistency conditions (5.7), (5.9) and (5.10).

By comparing these with ours, we find that the results in [19] are equivalent to ours, since the difference is absorbed by the normalization of the primary fields. To see this, we first note that, from the two-point functions $B(j)$ and $B(j)^T$, our Φ_j and the primary fields Φ_j^T in [19] are related by

$$\Phi_j = E(j) \Phi_j^T, \quad (6.9)$$

with

$$E(j) = \left(\frac{B(j)}{B(j)^T} \right)^{\frac{1}{2}} = -\mathcal{N}^{\frac{1}{2}} \pi^2 b^4 \left(\frac{b^2}{\pi k} \right)^j \Delta(b^2)^{2j+1} \Delta(-(2j+1)b^2). \quad (6.10)$$

⁶ We put an extra minus sign in the original expression in [19]. This sign is needed because j_3 is taken to zero from the positive real axis in [19] to obtain the two-point function from the three-point function.

This rescaling is consistent with the relation between $A(j)$ and $A(j)^T$:

$$A(j) = A(j)^T E(j) E(-j-1). \quad (6.11)$$

Furthermore, the three-point functions satisfy

$$D(j_a) = \left(\frac{1}{\pi^4 \mathcal{N}} \right)^{\frac{1}{2}} D(j_a)^T \prod_{a=1}^3 E(j_a). \quad (6.12)$$

Thus, with the choice of the normalization factor \mathcal{N} ,

$$\mathcal{N} = \frac{1}{\pi^4}, \quad (6.13)$$

the two results are in complete agreement including the numerical coefficients.

In addition, the relation between the normalizations in [18] and [19] has been discussed in [19]. From (6.9), we find that our normalization is essentially the same as that in [18].⁷ For generic j_a , the choice of these two normalizations is irrelevant to the pole structure. However, it is relevant in some cases.

7 DISCUSSION

Using the path integral approach, we discussed the correlation functions of the primary fields in the $SL(2, \mathbf{C})/SU(2)$ WZW model which corresponds to the string theory on the Euclidean AdS_3 . Because of the non-compactness of $SL(2, \mathbf{C})/SU(2) = H_3^+$, a careful definition of the correlation functions was necessary. We argued that the calculation for generic primary fields is reduced to that of Gawędzki for certain invariants with non-negative half-integral spins. The point was the $SL(2, \mathbf{C})$ projection for Φ_j in section 4.2 and the analytic continuation in the spin j . Regarding the latter, there still remain subtleties and hence we may need further discussions for a rigorous treatment. We then carried out an explicit calculation of the two- and three-point functions and obtained their closed forms. The three-point function was reduced to the supergravity result in the semi-classical limit. Furthermore, by an appropriate change of the normalization of the primary fields, we found an exact agreement with the results by Teschner using the bootstrap approach. Notice that a mere analytic continuation of Gawędzki's correlation functions C_2, C_3 does not reproduce the Teschner's results. The coefficients a_2, a_3 which were derived from the $SL(2, \mathbf{C})$ projection were important.

As discussed in the introduction, the H_3^+ WZW model has applications in various directions. The exact result of the correlation functions will be used for these investigations. Some applications are found in [36, 11]. In particular, it will serve as the starting point for

⁷ The normalization in [18] is not completely fixed.

the precise understanding of the AdS/CFT correspondence beyond the supergravity approximation. It is also interesting to apply our formalism to the correlation functions of other fields, such as the energy-momentum tensor of the boundary CFT.

As for the H_3^+ WZW model itself, it is important to study the fusion rules and the issue of the factorization in order to know the true spectrum of the model. From (2.14), the Hilbert space of the model consists of the principal continuous series. Thus, the states in this series may give the complete basis when one factorizes the four-point functions. On the other hand, since the spins take continuous values, the poles in the three-point function may contribute to the fusion rules and the states in other representations may appear. The role of such states seems to be similar to that of the non-normalizable states in the Liouville theory. These issues have also been discussed in [18, 19]. In the case of the $SL(2, \mathbf{R})$ WZW model corresponding to the Lorentzian AdS_3 , it has been argued that the winding modes play an important role [11][29][30] (see also [31][32]). It is not clear how to incorporate such modes in our formalism.

In our formalism or that in [18, 19], it seems difficult to push the calculation to the higher point functions though it is possible in principle. The free field approach discussed in [20, 10] (and the free field approach to \widehat{sl}_2) is certainly a powerful tool for this purpose. As discussed in section 3, the path integral approach gives expressions which look very similar to those in the free field approach. This implies that, when appropriately treated, the free field approach might be used in the region besides near the boundary of H_3^+ . Thus, it will be useful to consider the precise connection between these two approaches.

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A INTEGRAL FORMULAS

In this appendix we collect useful integral formulas. The first one is the Dotsenko-Fateev formula [33], given by

$$\begin{aligned}
J_n(\alpha, \beta, \rho) &= \int \prod_{i=1}^n d^2 y_i |y_i|^{2\alpha} |y_i - 1|^{2\beta} \prod_{i < j} |y_i - y_j|^{-4\rho} \\
&= n! \pi^n \left(\frac{\Gamma(1+\rho)}{\Gamma(-\rho)} \right)^n \prod_{l=1}^n \frac{\Gamma(-l\rho)}{\Gamma(1+l\rho)} \\
&\quad \times \prod_{l=0}^{n-1} \frac{\Gamma(1+\alpha-l\rho) \Gamma(1+\beta-l\rho) \Gamma(-1-\alpha-\beta+(n-1+l)\rho)}{\Gamma(-\alpha+l\rho) \Gamma(-\beta+l\rho) \Gamma(2+\alpha+\beta-(n-1+l)\rho)}.
\end{aligned} \tag{A.1}$$

Setting $n = 1$ and $\rho = 0$ in the above, we obtain the second one,

$$\int d^2 z |z|^{2\alpha} |z - 1|^{2\beta} = \pi \Delta(1+\alpha) \Delta(1+\beta) \Delta(-1-\alpha-\beta), \tag{A.2}$$

where $\Delta(x) = \Gamma(x)/\Gamma(1-x)$.

B Υ -FUNCTION

Here, we give the definition of the Υ -function and its basic properties. The function $\Upsilon(x)$ is defined by [34] (see also [35])

$$\log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right], \tag{B.1}$$

with $Q = b + 1/b$. This integral converges in the strip $0 < \text{Re } x < Q$. For other x , $\Upsilon(x)$ is defined through the functional relations

$$\begin{aligned}
\Upsilon(x+b) &= \Delta(bx) b^{1-2bx} \Upsilon(x), \\
\Upsilon\left(x + \frac{1}{b}\right) &= \Delta\left(\frac{x}{b}\right) b^{\frac{2x}{b}-1} \Upsilon(x), \\
\Upsilon(Q-x) &= \Upsilon(x).
\end{aligned} \tag{B.2}$$

From these relations, one finds that $\Upsilon(x)$ has zeros at

$$x = -mb - \frac{n}{b}, \quad (m+1)b + \frac{n+1}{b}, \quad m, n \in \mathbf{Z}_{\geq 0}. \tag{B.3}$$

This function can be used in the analytic continuation of the the Dotsenko-Fateev integral J_n in (A.1) with respect to n . To see this, we first denote a Dotsenko-Fateev integral by $I_m(\alpha_a)$:

$$\begin{aligned}
I_m(\alpha_1, \alpha_2, \alpha_3) &= J_m(-2b\alpha_1, -2b\alpha_2, b^2) \\
&= \Gamma(m+1) \left(\pi \Delta(1+b^2) \right)^m \prod_{l=1}^m \Delta(-lb^2) \prod_{l=0}^{m-1} \prod_{a=1}^3 \Delta^{-1}(2b\alpha_a + lb^2),
\end{aligned} \tag{B.4}$$

where α_3 is given by

$$\Sigma \equiv \sum_{i=1}^3 \alpha_i = Q - mb. \quad (\text{B.5})$$

Using (B.2), this can be rewritten as [34, 37]

$$\begin{aligned} I_m(\alpha_a) &= \Gamma(m+1) \left(-\pi b^{2-2b^2} \Delta(b^2) \right)^m \frac{\Upsilon'(0)}{\Upsilon'(-mb)} \prod_{i=1}^3 \frac{\Upsilon(2\alpha_i)}{\Upsilon(\Sigma - 2\alpha_i)} \\ &= \left(\pi b^{-2b^2} \Delta(b^2) \right)^m \frac{\Upsilon'(0)}{\Gamma(m+1)\Upsilon((m+1)b)} \prod_{i=1}^3 \frac{\Upsilon(2\alpha_i)}{\Upsilon(\Sigma - 2\alpha_i)}. \end{aligned} \quad (\text{B.6})$$

Here, $\Upsilon'(x) = d\Upsilon/dx$ and we have used the relation

$$\Upsilon'(-mb) = (-1)^m b^{2m} \Upsilon((m+1)b) \Gamma(m+1)^2. \quad (\text{B.7})$$

In particular, we have

$$\Upsilon'(0) = \Upsilon(b). \quad (\text{B.8})$$

REFERENCES

- [1] P.M. Petropoulos, “*String theory on AdS_3 : some open questions*”, [hep-th/9908189](#).
- [2] J.M. Maldacena, “*The large N limit of superconformal field theories and supergravity*”, *Adv. Theor. Math. Phys.* **2** (1998) 231-252, *Int. J. Theor. Phys.* **38** (1999) 1113-1133, [hep-th/9711200](#).
- [3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “*Gauge theory correlators from non-critical string theory*”, *Phys. Lett.* **B428** (1998) 105-114, [hep-th/9802109](#).
- [4] E. Witten, “*Anti de Sitter space and holography*”, *Adv. Theor. Math. Phys.* **2** (1998) 253-291, [hep-th/9802150](#).
- [5] A. Giveon, D. Kutasov and N. Seiberg, “*Comments on string theory on AdS_3* ”, *Adv. Theor. Math. Phys.* **2** (1998) 733-780, [hep-th/9806194](#).
- [6] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, “*String theory on AdS_3* ”, *JHEP* **9812** (1998) 026, [hep-th/9812046](#).
- [7] D. Kutasov and N. Seiberg, “*More comments on string theory on AdS_3* ”, *JHEP* **9904** (1999) 008, [hep-th/9903219](#).
- [8] N. Seiberg and E. Witten, “*The $D1/D5$ system and singular CFT*”, *JHEP* **9904** (1999) 017, [hep-th/9903224](#).

- [9] I. Bars, C. Deliduman and D. Minic, “*String theory on AdS_3 revisited*”, [hep-th/9907087](#)
- [10] G. Giribet and C.A. Núñez, JHEP **9** (9) 11(1999)031, “*Interacting strings on AdS_3* ”, JHEP **9911** (1999) 031, [hep-th/9909149](#).
- [11] J. Maldacena and H. Ooguri, “*Strings in AdS_3 and $SL(2, \mathbf{R})$ WZW model : I*”, [hep-th/0001053](#).
- [12] A. Kato and Y. Satoh, “*Modular invariance of string theory on AdS_3* ”, [hep-th/0001063](#).
- [13] A.L. Larsen and N. Sánchez, “*Quantum coherent string states in AdS_3 and $SL(2, \mathbf{R})$ WZWN model*”, [hep-th/0001180](#).
- [14] I. Pesando, “*Some remarks on the free fields realization of the bosonic string on AdS_3* ”, [hep-th/0003036](#).
- [15] Y. Hikida, K. Hosomichi, Y. Sugawara, “*String theory on AdS_3 as discrete light-cone Liouville theory*”, [hep-th/0005065](#).
- [16] Z. Haba, “*Correlation functions of σ -fields with values in a hyperbolic space*”, Int. J. Mod. Phys. **A4** (1989) 267-286.
- [17] K. Gawędzki, “*Quadrature of conformal field theories*”, Nucl. Phys. **B328** (1989) 733-752;
“*Non-compact WZW conformal field theories*”, NATO ASI: Cargese 1991: 0247-274, [hep-th/9110076](#).
- [18] J. Teschner, “*On structure constants and fusion rules in the $SL(2, \mathbf{C})/SU(2)$ WZNW model*”, Nucl. Phys. **B546** (1999) 390-422, [hep-th/9712256](#);
“*The mini-superspace limit of the $SL(2, \mathbf{C})/SU(2)$ -WZNW model*”, Nucl. Phys. **B546** (1999) 369-389, [hep-th/9712258](#).
- [19] J. Teschner, “*Operator product expansion and factorization in the H_3^+ -WZNW model*”, Nucl. Phys. **B571** (2000) 555-582, [hep-th/9906215](#).
- [20] J.-S. Caux, N. Taniguchi and A.M. Tsvelik, “*Termination of multifractal behaviour for critical disordered Dirac fermions*”, Phys. Rev. Lett. **80** (1998) 1276, [cond-mat/9711109](#);
“*Disordered Dirac fermions: multifractality termination and logarithmic conformal field theories*”, Nucl. Phys. **B525** (1998) 671-696, [cond-mat/9801055](#).
- [21] W. Mück and K.S. Viswanathan, “*Conformal field theory correlators from classical scalar field theory on AdS_{d+1}* ”, Phys. Rev. **D58** (1998) 041901, [hep-th/9804035](#).
- [22] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, “*Correlation functions in the CFT_d/AdS_{d+1} correspondence*”, Nucl. Phys. **B546** (1999) 96, [hep-th/9804058](#).
- [23] T. Kawano and K. Okuyama, “*Spinor exchange in AdS_{d+1}* ”, Nucl. Phys. **B565** (2000) 427-444, [hep-th/9905130](#).

- [24] I.M. Gel'fand, M.I. Graev, N.Ya. Vilenkin, “*Generalized functions*”, vol. 5, Academic Press, 1966.
- [25] I.M. Gel'fand and G.E. Shilov, “*Generalized functions*”, vol. 1, Academic Press, 1964.
- [26] A. Gerasimov, A. Morozov, M. Olshanetsky, A. Marshakov and S. Shatashvili, “Wess-Zumino-Witten model as a theory of free fields”, *Int. J. Mod. Phys.* **A5** (1990) 2495.
- [27] R. Kallosh, “Green-Schwarz action and loop calculations for superstring”, *Int. J. Mod. Phys.* **A3** (1988) 1943.
- [28] M. Goulian and M. Li, “Correlation functions in Liouville theory”, *Phys. Rev. Lett.* **66** (1991) 2051-2055.
- [29] J. Maldacena, H. Ooguri and J. Son, “Strings in AdS_3 and the $SL(2, \mathbf{R})$ WZW model. II: Euclidean black hole”, [hep-th/0005183](#).
- [30] G. Giribet and C. Núñez, “Aspects of the free field description of string theory on AdS_3 ”, *JHEP* **0006** (2000) 033, [hep-th/0006070](#).
- [31] M. Henningson, S. Hwang, P. Roberts and B. Sundborg, “Modular invariance of $SU(1,1)$ strings”, *Phys. Lett.* **B267** (1991) 350.
- [32] Y. Satoh, “Ghost-free and modular invariant spectra of a string in $SL(2, \mathbf{R})$ and three-dimensional black hole geometry”, *Nucl. Phys.* **B513** (1998) 213, [hep-th/9705208](#).
- [33] V.I.S. Dotsenko and V.A. Fateev, “Conformal algebra and multipoint correlation function in 2D statistical models”, *Nucl. Phys.* **B251**[FS12] (1984) 312-348;
“Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge $c \leq 1$ ”, *Nucl. Phys.* **B251**[FS13] (1985) 691-734.
- [34] A.B. Zamolodchikov and Al.B. Zamolodchikov, “Structure constants and conformal bootstrap in Liouville field theory”, *Nucl. Phys.* **B477** (1996) 577-605, [hep-th/9506136](#).
- [35] H. Dorn and H.J. Otto, “Two and three point functions in Liouville theory”, *Nucl. Phys.* **B429** (1994) 375-388, [hep-th/9403141](#).
- [36] A. Giveon and D. Kutasov, “Little string theory in a double scaling limit”, *JHEP* **9910** (1999) 034, [hep-th/9909110](#);
“Comments on double scaled little string theory”, *JHEP* **0001** (2000) 023, [hep-th/9911039](#).
- [37] L. O’Raifeartaigh, J.M. Pawłowski and V.V. Sreedhar, “Duality in quantum Liouville theory”, *Ann. Phys. (N.Y.)* **277** (1999) 117-143, [hep-th/9811090](#).